Exercise 52

Find y' if $x^y = y^x$.

Solution

Differentiate both sides with respect to x, using logarithms to bring the exponents down.

$$\frac{d}{dx}(x^y) = \frac{d}{dx}(y^x)$$

$$\frac{d}{dx}\left(e^{\ln x^y}\right) = \frac{d}{dx}\left(e^{\ln y^x}\right)$$

$$\frac{d}{dx}\left(e^{y\ln x}\right) = \frac{d}{dx}\left(e^{x\ln y}\right)$$

$$e^{y\ln x}\cdot\frac{d}{dx}(y\ln x) = e^{x\ln y}\cdot\frac{d}{dx}(x\ln y)$$

$$e^{y\ln x}\cdot\left\{\left[\frac{d}{dx}(y)\right]\ln x + y\left[\frac{d}{dx}(\ln x)\right]\right\} = e^{x\ln y}\cdot\left\{\left[\frac{d}{dx}(x)\right]\ln y + x\left[\frac{d}{dx}(\ln y)\right]\right\}$$

$$e^{y\ln x}\cdot\left[\left(\frac{dy}{dx}\right)\ln x + y\left(\frac{1}{x}\right)\right] = e^{x\ln y}\cdot\left\{(1)\ln y + x\left[\frac{1}{y}\cdot\frac{d}{dx}(y)\right]\right\}$$

$$e^{y\ln x}\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right) = e^{x\ln y}\left(\ln y + \frac{x}{y}\frac{dy}{dx}\right)$$

$$e^{\ln x^y}\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right) = e^{\ln y^x}\left(\ln y + \frac{x}{y}\frac{dy}{dx}\right)$$

$$x^y\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right) = y^x\left(\ln y + \frac{x}{y}\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}x^y\ln x + \frac{x^y y}{x} = y^x\ln y + \frac{y^x x}{y}\frac{dy}{dx}$$

Multiply both sides by xy to wipe out the fractions.

$$\frac{dy}{dx}x^{y+1}y\ln x + x^y y^2 = y^{x+1}x\ln y + y^x x^2\frac{dy}{dx}$$

Solve for dy/dx.

$$\frac{dy}{dx}x^{y+1}y\ln x - y^x x^2\frac{dy}{dx} = y^{x+1}x\ln y - x^y y^2$$

$$(x^{y+1}y\ln x - y^x x^2)\frac{dy}{dx} = y^{x+1}x\ln y - x^y y^2$$

$$\frac{dy}{dx} = \frac{y^{x+1}x\ln y - x^y y^2}{x^{y+1}y\ln x - y^x x^2}$$